

On the Papapetrou field in vacuum

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Abstract. In this paper we study the electromagnetic fields generated by a Killing vector field in vacuum space-times (*Papapetrou fields*). The motivation of this work is to provide new tools for the resolution of Maxwell's equations as well as for the search, characterization, and study of exact solutions of Einstein's equations. The first part of this paper is devoted to an algebraic study in which we give an explicit and covariant procedure to construct the principal null directions of a Papapetrou field. In the second part, we focus on the main differential properties of the principal directions, studying when they are geodesic, and in that case we compute their associated optical scalars. With this information we get the conditions that a principal direction of the Papapetrou field must satisfy in order to be aligned with a multiple principal direction of the Weyl tensor in the case of algebraically special vacuum space-times. Finally, we illustrate this study using the Kerr, Kasner and *pp waves* space-times.

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1. Introduction

In a relevant and historic paper, Papapetrou [1] pointed out that a Killing vector field (KVF), say ξ , can be considered as the vector potential generating an electromagnetic field with current $j^a = R^a_b \xi^b$ and satisfying a covariant version of the Lorentz gauge. As an immediate consequence, in vacuum space-times this electromagnetic field, which we will call here the *Papapetrou field*, satisfies the Maxwell equations in the absence of electromagnetic sources.

The Papapetrou fields have been used in the search and study of exact solutions of Einstein's field equations (see e.g. [2, 3]), and in the study of black holes under external electromagnetic fields [4]. In particular, it turns out that the Kerr-Newman electromagnetic field is a Papapetrou field generated by the timelike KVF of the Kerr metric (for more information see references [4, 5]).

The applications and motivation of this paper concern with issues in which the Papapetrou field plays a central role. The first one is the resolution of Maxwell's equations in a curved space-time, which is a subject of special relevance for instance

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in the study of electromagnetic perturbations of black holes (an exhaustive account is given in [6]). When we write Maxwell equations in a null basis, using the Newman-Penrose formalism [7] (see also [6, 8]), we get a set of first-order partial differential equations for the complex components of the electromagnetic field, namely Φ_A ($A = 0, 1, 2$). The integrability conditions for the complex component Φ_1 (the *conditional* system) form in general a third-order differential system (with respect to the components Φ_0 and Φ_2), however, as it was shown in [9], if we choose the Newman-Penrose basis to be adapted to the principal null directions (or null eigendirections) of any particular regular solution of Maxwell's equations, the resulting conditional system is of second order. This is an important fact in order to integrate Maxwell's equations. For instance, as it was remarked in [9], this is just what happens with the Teukolsky-Press relations [10, 11], which were the starting point for the integration of Maxwell's equations in perturbed Kerr space-times (see [6] and references therein). In [9], it is shown that the Teukolsky-Press relations, completed with an additional equation, constitute a conditional system of second order for Maxwell's equations (in the variables Φ_0 and Φ_2). This is due to the fact that they were written using a Newman-Penrose basis adapted to the principal directions of the Kerr space-time, which are also the principal directions of a regular electromagnetic field (see also [9]). These facts emphasize the importance of knowing a particular regular solution of Maxwell's equations. In general space-times we do not have a method to construct such a solution, but in the case of vacuum space-times possessing a KVF, a particular solution is given by the Papapetrou field. Hence, part of this paper is devoted to make an exhaustive algebraic study of these electromagnetic fields, giving the procedure to construct a Newman-Penrose basis in which the integrability conditions of the Maxwell equations are directly second-order partial differential equations. For the sake of completeness we will also treat the case of singular Papapetrou fields, which appear in relevant space-times like the well-known *pp waves*.

On the other hand, Papapetrou fields provide us a link between the Killing symmetries and the algebraic structure of the space-time, which is a subject treated only occasionally in the literature (see [8], Chapter 33). This relationship can be found by studying the alignment of the principal directions of the Papapetrou field with those of the gravitational field, i.e., of the Riemann tensor (which in vacuum reduces to the Weyl tensor). In this sense, a remarkable example is the Kerr metric [12]. In [13, 14] it is shown that the principal directions of the Papapetrou field associated with the timelike KVF coincide with the two repeated principal directions of the space-time (it is Petrov type D). This link between symmetries and algebraic structure provides a powerful tool for the study and search of vacuum gravitational fields. As an example, a new characterization of the Kerr metric can be found. In a recent work by Mars [14], it is shown that the Kerr metric is the only stationary asymptotically-flat vacuum space-time in which the aforementioned alignment is given. This characterization of the Kerr metric is obtained everywhere, not only where the Killing is timelike. Here, we will see an alternative and simple way of characterizing the Kerr metric by using our formalism and the connection with the eigenray formalism of Perjés [15].

These facts motivate the study of the differential properties of the principal directions of the Papapetrou field. In particular, the study of under which conditions they are geodesic and shear-free will tell us, via the well-known Goldberg-Sachs theorem [16], when a principal direction of the Papapetrou field is aligned with a multiple principal direction of an algebraically special vacuum gravitational field. As we will see in the development of the paper, these properties, and other properties related with the optical scalars of the principal directions, can be determined only in terms of the principal direction itself and the Ernst potential associated with the KVF under consideration. This result is relevant in the sense that it provides a way of making interesting *Ansätze* for the search of new solutions of Einstein's equations which, in the light of the example of the Kerr metric and other examples that will be given in this paper, can be physically well motivated.

The plan of this paper is the following: in Section 2, we give some general properties of antisymmetric tensors (2-forms), which will be needed for the development of the next sections. In Section 3, we introduce the Papapetrou field associated with a KVF. Then, we make an exhaustive study of its algebraic structure, for which we shall distinguish between regular and singular fields. In the regular case, we will introduce a complex 1-form, proportional to the differential of the Ernst potential, which will play a central role in our study. Then, we will determine explicitly (in terms of quantities defined only from the KVF and the metric) the eigenvalues, the principal null directions, and the orthogonal 2-planes to these directions (the Maxwellian structure). In the singular case we determine the only principal null direction. In Section 4, we use this algebraic study to analyze the main differential properties of the principal direction(s). We will give the conditions for a principal direction to be geodesic, and for that case we will give the expression for the optical scalars, which can be written in terms of the complex 1-form mentioned above and the principal direction. An illustration of how this study works is given in Section 5, where some examples are discussed, in particular the case of the Kerr metric [12] is treated with great detail. In Section 6, we discuss the main results and consequences of this work, as well as possible extensions. Finally, in Appendix A we summarize the known results about the case of a null Killing vector, and in Appendix B we give the connection with the *eigenray* formalism of Perjés [15]. Through this paper we will follow the notation and conventions of [8] unless otherwise stated.

2. Algebraic structure of a 2-form

In this section we review briefly the algebraic structure of a 2-form \mathbf{F} (an antisymmetric tensor), describing an explicit and covariant procedure to construct the null principal direction(s) (eigendirections). First, we deal with the regular (non-singular) case, that is, when at least one of the two invariants is not zero:

$$\tilde{F}^{ab}\tilde{F}_{ab} \neq 0, \quad \tilde{F}_{ab} \equiv F_{ab} + i * F_{ab}, \quad *F_{ab} \equiv \frac{1}{2}\eta_{abcd}F^{cd},$$

where $*$ and \sim denote the dual and self-dual operations respectively, and η_{abcd} are the components of the volume 4-form. At the end of this section we will extend the procedure to the singular case.

A regular 2-form \mathbf{F} can be completely characterized in terms of its eigenvalues (α, β) and null principal directions $(\mathbf{k}, \mathbf{\ell})$. In fact, it can be written as follows

$$\mathbf{F} = \alpha \mathbf{G} - \beta * \mathbf{G}, \quad \mathbf{G} \equiv \mathbf{k} \wedge \mathbf{\ell}, \quad k^a \ell_a = -1. \quad (1)$$

Note that \mathbf{G} is a 2-form of rank 2 [$\det(\mathbf{G}) = 0$]. Moreover, the corresponding dual and self-dual 2-forms $*\mathbf{F}$ and $\tilde{\mathbf{F}}$ are given by

$$*\mathbf{F} = \beta \mathbf{G} + \alpha * \mathbf{G}, \quad \tilde{\mathbf{F}} = (\alpha + i\beta) \tilde{\mathbf{G}}, \quad (2)$$

from where we find the following relation between the invariants of \mathbf{F} and its eigenvalues

$$\tilde{F}^{ab} \tilde{F}_{ab} = -4(\alpha + i\beta)^2. \quad (3)$$

Given a 2-form \mathbf{F} , this relation allows us to find the eigenvalues (α, β) , and from (2) we can find the singular 2-form \mathbf{G} . On the other hand, for each 2-form \mathbf{F} we can construct the following symmetric tensor

$$T_{ab} = \frac{1}{2} (F_a^c F_{bc} + *F_a^c *F_{bc}). \quad (4)$$

When \mathbf{F} represents an electromagnetic field, this tensor is the energy-momentum tensor. In the case of a regular 2-form, using (1,2) we find an alternative expression for T_{ab}

$$T_{ab} = (\alpha^2 + \beta^2) [k_{(a} \ell_{b)} + m_{(a} \bar{m}_{b)}], \quad (5)$$

where \mathbf{m} is a complex vector orthogonal to the 2-planes generated by the principal directions \mathbf{k} and $\mathbf{\ell}$, and such that $\{\mathbf{k}, \mathbf{\ell}, \mathbf{m}, \bar{\mathbf{m}}\}$ is a Newman-Penrose basis. Therefore, this basis is adapted to the Maxwellian structure of the 2-form \mathbf{F} . We can write the dual of \mathbf{G} in terms of \mathbf{m} and its complex conjugate $\bar{\mathbf{m}}$ as $*\mathbf{G} = i\mathbf{m} \wedge \bar{\mathbf{m}}$. Note that expression (5) shows the 2+2 decomposition of the energy-momentum tensor (equivalently of the Ricci tensor) of a regular 2-form, or in other words, that the Segré type is $[(1 \ 1) \ (1 \ 1)]$ (see for instance [8]).

The problem of determining the principal directions of a 2-form has been treated by Coll and Ferrando [17]. To that end they introduced the so-called *concomitant* of a 2-form

$$\mathcal{F}_{ab} \equiv \alpha F_{ab} + \beta *F_{ab} - T_{ab} + \frac{1}{2}(\alpha^2 + \beta^2)g_{ab}. \quad (6)$$

Using this tensor, the principal null directions are given by

$$K_a = \mathcal{F}_{ab} U^b, \quad L_a = \mathcal{F}_{ba} U^b, \quad (7)$$

where \mathbf{U} is an arbitrary timelike vector field. We can understand this property by rewriting the tensor \mathcal{F} in the following form

$$\mathcal{F}_{ab} = (\alpha^2 + \beta^2) (G_{ab} + G_a^c G_{cb}) = -2(\alpha^2 + \beta^2) \ell_a k_b, \quad (8)$$

where we have used (1-4). As we can see from this expression, \mathbf{K} and \mathbf{L} are parallel to the null vector fields \mathbf{k} and $\mathbf{\ell}$ respectively [see expressions (1)]. It is important to point

out that this method provides a covariant way of finding the principal directions of a 2-form.

On the other hand, in order to determine the 2-planes orthogonal to the principal directions, we can introduce another tensor

$$\mathcal{F}_{ab}^\perp \equiv i(\beta F_{ab} - \alpha * F_{ab}) + T_{ab} + \frac{1}{2}(\alpha^2 + \beta^2)g_{ab}. \quad (9)$$

From this complex tensor we can find a complex vector $\bar{\mathbf{m}}$ describing the orthogonal 2-planes. To that end, we have to contract (9) with any spacelike vector field linearly independent from \mathbf{k} and ℓ , and normalize the result ($m^a \bar{m}_a = 1$). We can show this fact by rewriting \mathcal{F}_{ab}^\perp in terms of the Newman-Penrose basis $\{\mathbf{k}, \ell, \mathbf{m}, \bar{\mathbf{m}}\}$

$$\mathcal{F}_{ab}^\perp = -(\alpha^2 + \beta^2)(i * G_{ab} + * G_a{}^c * G_{cb}) = -2(\alpha^2 + \beta^2)m_a \bar{m}_b,$$

Note that \mathbf{m} is fixed up to a factor e^{iC} , where C is an arbitrary real scalar.

To sum up, we can construct the Maxwellian structure associated with a regular 2-form from tensors (6) and (9).

In the singular case, the self-dual 2-form $\tilde{\mathbf{F}}$ can always be written as follows (see [8])

$$\tilde{F}_{ab} = 4\Phi k_{[a} m_{b]}, \quad k^a m_a = 0,$$

where \mathbf{m} is some normalized null complex vector field ($m^a m_a = 0$ and $m^a \bar{m}_a = 1$) and Φ is a complex scalar field. As is well-known, \mathbf{k} is a multiple eigenvector with zero eigenvalue. In addition, the energy-momentum tensor (4) now takes the following form

$$T_{ab} = 2\Phi \bar{\Phi} k_a k_b,$$

that is, the Segré type is $[(1 \ 1, 2)]$. We can use the previous procedure to find explicitly the principal direction only by considering the fact that the eigenvalues are identically zero ($\alpha = \beta = 0$). Therefore, the tensor \mathcal{F}_{ab} is simply $-T_{ab}$, showing that there is only one principal direction (\mathbf{K} and \mathbf{L} coincide), which is determined by

$$K_a = T_{ab} U^b,$$

where U^a is an arbitrary timelike vector field. In that case, the tensor \mathcal{F}_{ab}^\perp does not provide any information since there is not a 2+2 structure, which is clear from the Segré type of the energy-momentum tensor.

3. The Papapetrou field

Let (V_4, g) be an arbitrary vacuum ($R_{ab} = 0$) space-time (see [18] for details) endowed with a one-dimensional isometry group generated by a non-null KVF ξ

$$\xi_{a;b} + \xi_{b;a} = 0. \quad (10)$$

That is, we will assume that ξ has a fixed character (timelike or spacelike) in an open domain of the space-time^{||}. In this domain, the electromagnetic field, *Papapetrou field*, generated by ξ is given by [1]

$$F_{ab} = (d\xi)_{ab} = \xi_{b;a} - \xi_{a;b} = 2\xi_{b;a}, \quad (11)$$

^{||} The case of a null KVF is of little interest since all the vacuum space-times admitting a null KVF are known, they can be found in [8] (section 21.4; see also Ref. [19] and Appendix A).

where we have used the Killing equations (10) and \mathbf{d} denotes exterior differentiation. Note that \mathbf{F} is defined up to a multiplicative constant. Moreover, taking into account the following relation for the second covariant derivatives of a KVF

$$\xi_{a;bc} = R_{abcd}\xi^d, \quad (12)$$

which comes from the Ricci identities, we can show that this 2-form F_{ab} satisfies the Maxwell equations in the absence of electromagnetic charge and current distributions

$$F_{[ab;c]} = 0, \quad F^{ab}{}_{;b} = 0. \quad (13)$$

In addition, from the definition of the Papapetrou field (11), we can see that the KVF ξ plays the role of the electromagnetic vector potential, and from the Killing equations (10), that it satisfies the covariant Lorentz condition $\xi^a{}_{;a} = 0$.

Some important quantities associated with a KVF are: the *norm*

$$N \equiv \xi^a \xi_a,$$

which we have assumed to be different from zero, the *orthogonal projector*

$$h_{ab} \equiv g_{ab} - \frac{1}{N} \xi_a \xi_b, \quad h_{ab} \xi^b = 0,$$

the gradient of the norm

$$\psi_a \equiv -(\mathbf{d}N)_a = -N_{;a}, \quad \psi^a \xi_a = 0,$$

and the *twist*

$$\omega_a \equiv - * (\xi \wedge \mathbf{d}\xi)_a = \eta_{abcd} \xi^b \xi^{c;d}, \quad \omega^a \xi_a = 0.$$

The twist has three independent components which are also contained in the following 2-form

$$W_{ab} = h_a{}^c h_b{}^d \xi_{[c;d]} = h_a{}^c h_b{}^d \xi_{c;d}, \quad W_{ab} \xi^b = 0, \quad (14)$$

usually called the *rotation* of ξ . The rotation is related to the twist through the expressions

$$\omega_a = \eta_{abcd} \xi^b W^{cd}, \quad W_{ab} = \frac{1}{2N} \eta_{abcd} \xi^c \omega^d.$$

In terms of these quantities, the Papapetrou field can be written in the following way

$$F_{ab} = \frac{2}{N} (\xi_{[a} \psi_{b]} - N W_{ab}), \quad *F_{ab} = \frac{2}{N} (\xi_{[a} \omega_{b]} + N M_{ab}), \quad (15)$$

where the definition of M_{ab} is

$$M_{ab} \equiv \frac{1}{2N} \eta_{abcd} \xi^c \psi^d.$$

On the other hand, it is well-known that in vacuum space-times the twist 1-form satisfies $\mathbf{d}\omega = 0$ (see [8] for instance), and therefore there is locally a function Ω so that $\omega = \mathbf{d}\Omega$. Then, we can introduce the Ernst potential (see [20, 8]) associated with ξ

$$\mathcal{E} = -N + i\Omega.$$

In terms of this potential, the self-dual Papapetrou field can be cast into the following form

$$\tilde{\mathbf{F}} = \frac{1}{N} (\boldsymbol{\xi} \wedge \boldsymbol{\epsilon})^\sim, \quad \boldsymbol{\epsilon} \equiv d\mathcal{E} = \boldsymbol{\psi} + i\boldsymbol{\omega} \quad (d\boldsymbol{\epsilon} = 0), \quad (16)$$

where we have defined $\boldsymbol{\epsilon}$ as the exterior derivative of the Ernst potential \mathcal{E} . In vacuum, this 1-form satisfies the following equation

$$\epsilon^a{}_{;a} + N^{-1} \epsilon^a \epsilon_a = 0, \quad (17)$$

which comes from Einstein's equations. Moreover, from (16) we realize that in the case of a timelike KVF, $\boldsymbol{\psi}$ and $\boldsymbol{\omega}$ are respectively proportional to the electric and magnetic fields measured by observers with 4-velocity $\mathbf{u} = (-N)^{-1/2} \boldsymbol{\xi}$.

We begin the process of construction of the principal directions of the Papapetrou field by the calculation of the eigenvalues α and β . Using (15) or (16) we get

$$\tilde{F}^{ab} \tilde{F}_{ab} = \frac{4}{N} \epsilon^a \epsilon_a = \frac{4}{N} \{ \psi^a \psi_a - \omega^a \omega_a + 2i \psi^a \omega_a \}.$$

And comparing this expression with (3) we find

$$\alpha = \frac{s}{\sqrt{2}} \sqrt{-x + \sqrt{x^2 + y^2}}, \quad \beta = -\frac{s}{\sqrt{2}} \sqrt{x + \sqrt{x^2 + y^2}}, \quad (18)$$

where x and y

$$x \equiv \frac{\psi^a \psi_a - \omega^a \omega_a}{N}, \quad y \equiv \frac{2\psi^a \omega_a}{N}, \quad (19)$$

are the two invariants of the Papapetrou field \mathbf{F} , and s is an arbitrary sign ($s^2 = 1$). This sign reflects the invariance of \mathbf{F} (1) under the change $(\mathbf{k}, \boldsymbol{\ell}, \alpha, \beta) \rightarrow (\boldsymbol{\ell}, \mathbf{k}, -\alpha, -\beta)$.

From (18), we find some interesting relationships between the eigenvalues and the invariants

$$\alpha^2 - \beta^2 = -x, \quad 2\alpha\beta = -y, \quad \alpha^2 + \beta^2 = \sqrt{x^2 + y^2}. \quad (20)$$

These relations together with (19) imply

$$\epsilon^a \epsilon_a = N(x + iy) = -N(\alpha + i\beta)^2. \quad (21)$$

On the other hand, from (4) we have the following expression from the energy-momentum tensor of the Papapetrou field (it is important to remark that this electromagnetic field is a test electromagnetic field)

$$T_{ab} = \frac{1}{N^2} \left\{ (\psi^c \psi_c + \omega^c \omega_c) \xi_a \xi_b + N(\psi_a \psi_b + \omega_a \omega_b) - 2N \xi_{(a} \phi_{b)} - \frac{1}{2} N (\psi^c \psi_c + \omega^c \omega_c) g_{ab} \right\}, \quad (22)$$

where

$$\phi_a \equiv 2W_{ab} \psi^b = -2M_{ab} \omega^b = -\frac{1}{N} \eta_{abcd} \xi^b \psi^c \omega^d.$$

In this situation, we are ready to construct the tensors \mathcal{F}_{ab} and \mathcal{F}_{ab}^\perp (6,9), and through (7) we can find explicit expressions for null vector fields tangent to the principal directions of the Papapetrou field. However, in order to carry out these calculations,

it is necessary to consider separate cases. First of all, we make a division into two groups depending on whether the Papapetrou field is regular ($\alpha + i\beta \neq 0$) or singular ($\alpha + i\beta = 0$). In the first case, we will determine completely the characteristic 2+2 structure of a regular 2-form, the Maxwellian structure. In the second case we will give the only one principal null direction.

3.1. Regular Papapetrou fields

We begin the study of regular Papapetrou fields by introducing a complex 1-form which will be of crucial importance for the purposes of this work. The definition of this 1-form is

$$\mathbf{E} \equiv \frac{\epsilon}{\alpha + i\beta} = \frac{1}{\alpha + i\beta}(\psi + i\omega), \quad \mathbf{E} \wedge d\mathbf{E} = 0. \quad (23)$$

As we can see, it is proportional to the exterior derivative of the Ernst potential and hence, it is integrable. Moreover, from (16) we find the following useful relationship between \mathbf{E} and the self-dual 2-form $\tilde{\mathbf{G}}$ [see Eq. (2)]

$$\tilde{\mathbf{G}} = \frac{1}{N} (\xi \wedge \mathbf{E})^\sim \iff E_a = -\tilde{G}_{ab} \xi^b.$$

On the other hand, \mathbf{E} satisfies the following equation

$$E^a{}_{;a} + \frac{E^a(\alpha + i\beta)_{,a}}{\alpha + i\beta} = \alpha + i\beta,$$

which follows from equation (17). On the other hand, when it is an exact 1-form, $d\mathbf{E} = 0$, either $\alpha + i\beta$ is constant or it is a function only of the Ernst potential.

The importance of this 1-form is twofold, first it will allow us to classify the different subcases that need a separate treatment, and second, it will serve to write our results in a compact form. Now, let us split \mathbf{E} into its real and imaginary parts

$$\mathbf{E} = \mathbf{E}_R + i\mathbf{E}_I. \quad (24)$$

From the definition (23) we can find the relations between $(\mathbf{E}_R, \mathbf{E}_I)$ and (ψ, ω)

$$\mathbf{E}_R = \frac{1}{\alpha^2 + \beta^2}(\alpha\psi + \beta\omega), \quad \mathbf{E}_I = \frac{1}{\alpha^2 + \beta^2}(-\beta\psi + \alpha\omega), \quad (25)$$

and the converse relations

$$\psi = \alpha\mathbf{E}_R - \beta\mathbf{E}_I, \quad \omega = \beta\mathbf{E}_R + \alpha\mathbf{E}_I.$$

From (23,24,21) we find the following important relation

$$E^a E_a = E_R^a E_{Ra} - E_I^a E_{Ia} + 2iE_R^a E_{Ia} = -N, \quad (26)$$

from where we deduce that \mathbf{E}_R and \mathbf{E}_I are orthogonal

$$E_R^a E_{Ia} = 0, \quad (27)$$

and also, a relation between their norms and the Killing norm (note that \mathbf{E} has a real norm)

$$E_R^a E_{Ra} - E_I^a E_{Ia} = -N. \quad (28)$$

From these properties we can derive some consequences concerning the character of the vector fields $\{\boldsymbol{\xi}, \mathbf{E}_R, \mathbf{E}_I, \boldsymbol{\phi}\}$. First, when $\boldsymbol{\xi}$ is timelike ($N < 0$), is obvious that \mathbf{E}_R and \mathbf{E}_I are spacelike, and taking into account that

$$\phi^a \phi_a = \frac{1}{N} [(\psi^a \omega_a)^2 - (\psi^a \psi_a)(\omega^b \omega_b)] = -\frac{1}{N}(\alpha^2 + \beta^2)^2 (E_R^a E_{Ra})(E_I^b E_{Ib}), \quad (29)$$

it is clear that $\boldsymbol{\phi}$ is also spacelike. On the contrary, when $\boldsymbol{\xi}$ is spacelike ($N > 0$), from (28) we have $E_I^a E_{Ia} = N + E_R^a E_{Ra}$ and then, \mathbf{E}_I cannot be timelike, otherwise \mathbf{E}_R would be also timelike, in contradiction with (27). Moreover, \mathbf{E}_I cannot be a null vector either, otherwise taking into account (28), which implies that $E_R^a E_{Ra} = -N < 0$, and (27), \mathbf{E}_R would be also a null vector and using again (28), $\boldsymbol{\xi}$ should be also a null vector, which contradicts our initial assumptions. Therefore, we have shown that \mathbf{E}_I *must be always spacelike* (unless it vanishes). In contrast, \mathbf{E}_R and $\boldsymbol{\phi}$ can be timelike, or spacelike, or null. Furthermore, when \mathbf{E}_R is a null vector, $\boldsymbol{\xi}$ is spacelike and $\boldsymbol{\phi}$ is also a null vector parallel to \mathbf{E}_R .

Now, let us introduce some vectors which are adapted to 2+2 decomposition of the energy-momentum tensor of an electromagnetic field. These vectors are \mathbf{E}_R , \mathbf{E}_I and

$$\mathbf{P} \equiv \frac{1}{N} [(E_R^a E_{Ra})\boldsymbol{\xi} + \mathbf{D}], \quad \mathbf{Q} \equiv \frac{1}{N} [(E_I^a E_{Ia})\boldsymbol{\xi} + \mathbf{D}], \quad (30)$$

where \mathbf{D} is a 1-form defined by $\mathbf{D} \equiv *(\boldsymbol{\xi} \wedge \mathbf{E}_R \wedge \mathbf{E}_I)$. In local coordinates the components of \mathbf{D} are

$$D_a = \eta_{abcd} \xi^b E_R^c E_I^d = \frac{1}{\alpha^2 + \beta^2} \eta_{abcd} \xi^b \psi^c \omega^d = -\frac{N}{\alpha^2 + \beta^2} \phi_a.$$

The scalar products between these 1-forms are given by

$$P^a P_a = -E_R^a E_{Ra}, \quad Q^a Q_a = E_I^a E_{Ia}, \quad (31)$$

$$P^a E_{Ra} = P^a E_{Ia} = P^a Q_a = Q^a E_{Ra} = Q^a E_{Ia} = 0,$$

and hence, they form in general an orthogonal basis (unless $E_R^a E_{Ra} = 0$ or $\mathbf{E}_I = 0$). The vectors \mathbf{P} and \mathbf{E}_R lie in the 2-planes generated by the principal directions. This fact can be seen from the following properties of \mathcal{F}_{ab}^\perp

$$\mathcal{F}_{ab}^\perp P^b = \mathcal{F}_{ab}^\perp E_R^b = 0, \quad \mathcal{F}_{ab}^\perp P^a = \mathcal{F}_{ab}^\perp E_R^a = 0. \quad (32)$$

On the other hand, \mathbf{Q} and \mathbf{E}_I determine in general (when $\mathbf{E}_I \neq 0$) the orthogonal 2-planes to the principal directions. This fact follows from the following relations

$$\mathcal{F}_{ab} Q^b = \mathcal{F}_{ab} E_I^b = 0, \quad \mathcal{F}_{ab} Q^a = \mathcal{F}_{ab} E_I^a = 0.$$

Moreover, since \mathbf{E}_I is always spacelike and taking into account (31), we deduce that both \mathbf{E}_I and \mathbf{Q} are spacelike.

In what follows we find explicit expressions for the principal directions of a regular Papapetrou field and determine the 2-planes orthogonal to these directions. Following the previous discussion we will distinguish between two subcases, depending on whether or not the norm of \mathbf{E}_R vanishes.

3.1.1. Generic case $E_R^a E_{Ra} \neq 0$: This case represents the generic situation. From (31) we deduce that either \mathbf{P} or \mathbf{E}_R must be timelike and hence, in order to obtain the principal directions we can use these vectors. However, in this case we can write \mathcal{F} in terms of the basis $\{\mathbf{P}, \mathbf{E}_R, \mathbf{Q}, \mathbf{E}_I\}$

$$\mathcal{F}_{ab} = \frac{\alpha^2 + \beta^2}{E_R^c E_{Rc}} (E_{Ra} + P_a)(E_{Rb} - P_b),$$

from where it follows that the principal directions of the Papapetrou field are determined by the following null vector fields

$$\mathbf{K} = \mathbf{P} - \mathbf{E}_R, \quad \mathbf{L} = \mathbf{P} + \mathbf{E}_R. \quad (33)$$

These expressions also show the fact that in general \mathbf{P} and \mathbf{E}_R determine the 2-planes generated by the principal directions, as it has been pointed out before [see expressions (32)]. In order to determine the orthogonal 2-planes we have to use the tensor (9), which in this generic case can be cast into the following form

$$\mathcal{F}_{ab}^\perp = \frac{\alpha^2 + \beta^2}{E_I^c E_{Ic}} (E_{Ia} - iQ_a)(E_{Ib} + iQ_b).$$

Then, the orthogonal 2-planes are determined by \mathbf{Q} and \mathbf{E}_I . Moreover, as is clear, this expression requires $E_I^a E_{Ia} \neq 0$, equivalent to $\mathbf{E}_I \neq 0$. In fact, the case $\mathbf{E}_I = 0$ needs a separate study, which is given below.

Finally, we want to stress the fact that the 2+2 structure is also manifested in the form of the energy-momentum tensor. In our case, we can see this feature by writing T_{ab} using the basis $\{\mathbf{P}, \mathbf{E}_R, \mathbf{Q}, \mathbf{E}_I\}$

$$T_{ab} = \frac{1}{2}(\alpha^2 + \beta^2) \left\{ \frac{1}{E_R^c E_{Rc}} (P_a P_b - E_{Ra} E_{Rb}) + \frac{1}{E_I^c E_{Ic}} (Q_a Q_b + E_{Ia} E_{Ib}) \right\}.$$

• *Particular case $\mathbf{E}_I = 0$:* Now we have $-\beta\psi + \alpha\omega = 0$ and then ψ and ω are linearly dependent, hence $\mathbf{D} = 0$. Moreover, this implies $\mathbf{P} = -\boldsymbol{\xi}$ and $\mathbf{E} = \mathbf{E}_R$, therefore we can use the previous expressions (33). Then, the principal null directions of the Papapetrou field are given by

$$\mathbf{K} = \boldsymbol{\xi} + \mathbf{E}_R, \quad \mathbf{L} = \boldsymbol{\xi} - \mathbf{E}_R. \quad (34)$$

On the other hand, the relations $E_I^a E_{Ia} = 0$ and $E_R^a E_{Ra} = -N$ [the last one comes from (28)] imply

$$\psi^a \psi_a + \omega^a \omega_a = -N(\alpha^2 + \beta^2),$$

where we have used (25). Combining the last equation with expressions (19,20) we arrive at the following results

$$\psi^a \psi_a = -N\alpha^2, \quad \omega^a \omega_a = -N\beta^2.$$

The peculiarity of this subcase is related with the structure of the orthogonal 2-planes. Now, although we can determine the 2-planes spanned by the principal null directions, we cannot determine explicitly the orthogonal 2-planes, since in this case we have $\mathbf{E}_I = \mathbf{Q} = 0$ [see equations (30)]. However, using the results of section 2 we can

construct the orthogonal 2-planes. We only need to contract \mathcal{F}_{ab}^\perp (9), which is now given by

$$\mathcal{F}_{ab}^\perp = -\frac{\alpha^2 + \beta^2}{N} \left\{ i\eta_{abcd} P^c E_R^d + P_a P_b - E_{Ra} E_{Rb} - N g_{ab} \right\}, \quad (35)$$

with any spacelike vector field independent of $\boldsymbol{\xi}$ and \mathbf{E}_R . Then, after normalization we find \mathbf{m} , which is defined up to a factor e^{iC} , where C is an arbitrary real scalar. Alternatively, given two vector fields, linearly independent and orthogonal to $\boldsymbol{\xi}$ and \mathbf{E}_R , we can construct from them the orthogonal 2-planes to the principal directions, and therefore we can find a vector \mathbf{m} .

3.1.2. Particular case $E_R^a E_{Ra} = 0$: Now, let us see what happens when \mathbf{E}_R is a null vector

$$E_R^a E_{Ra} = 0 \quad \implies \quad E_I^a E_{Ia} = N. \quad (36)$$

Following the discussion given above on the norm of \mathbf{E}_R and \mathbf{E}_I , it turns out that this case can only take place when $\boldsymbol{\xi}$ is spacelike ($N > 0$).

On the other hand, using (25), equations (28) and (36) imply

$$\psi^a \psi_a + \omega^a \omega_a = N(\alpha^2 + \beta^2).$$

Combining this result with (19,20) we get expressions for the norms of $\boldsymbol{\psi}$ and $\boldsymbol{\omega}$

$$\psi^a \psi_a = N\beta^2, \quad \omega^a \omega_a = N\alpha^2,$$

then, both are spacelike. Moreover, from (19,20) we can also deduce the following result

$$(\psi^a \psi_a)(\omega^a \omega_a) = (\psi^a \omega_a)^2.$$

This is an important relation because it means that either $\boldsymbol{\psi}$ and $\boldsymbol{\omega}$ are parallel or one of them is a null vector orthogonal to the other one. The first possibility implies that $\mathbf{E}_R = 0$, which will be treated later as an special subcase. The second possibility leads to one of the following two excluding situations: the first one is given by

$$\psi^a \psi_a = \psi^a \omega_a = \beta = 0, \quad \mathbf{E}_R = \frac{1}{\alpha} \boldsymbol{\psi}, \quad \mathbf{E}_I = \frac{1}{\alpha} \boldsymbol{\omega}, \quad (37)$$

and the second one by

$$\omega^a \omega_a = \psi^a \omega_a = \alpha = 0, \quad \mathbf{E}_R = \frac{1}{\beta} \boldsymbol{\omega}, \quad \mathbf{E}_I = -\frac{1}{\beta} \boldsymbol{\psi}. \quad (38)$$

In both situations (37,38), equation (29) implies that $\boldsymbol{\phi}$ is a null vector field parallel to \mathbf{E}_R , and obviously, the same happens with \mathbf{D} .

With regard to the principal directions, in this case we cannot determine completely the 2-planes spanned by the principal null directions, since \mathbf{P} and \mathbf{E}_R are not linearly independent ($\mathbf{P} = N^{-1} \mathbf{D}$ is also a null vector parallel to \mathbf{E}_R). From (33) we can see that \mathbf{E}_R is one of the principal directions, but we cannot determine the other one from these expressions. This is due to the fact that we have not a timelike vector available.

Therefore, it is convenient to follow the procedure explained in Sec. 2. First, we have to construct the tensor \mathcal{F}_{ab} (6). In this case, we can find \mathcal{F}_{ab} through the expression (8) and taking into account that now we can write

$$G_{ab} = \frac{1}{N} \eta_{abcd} E_I^c Q^d. \quad (39)$$

Finally, using an arbitrary timelike vector and expressions (7) we will find the remaining principal direction of the Papapetrou field.

We finish this case with the study of the following remaining subcase:

• *Particular subcase $\mathbf{E}_R = 0$* : The vanishing of $\mathbf{E}_R = 0$ is equivalent to the relation $\alpha\psi + \beta\omega = 0$, which tells us that ψ and ω are linearly dependent. The difference that appears between this subcase and the general case $E_R^a E_{Ra} = 0$ is that now we cannot determine explicitly any of the two principal directions, since now we have $\mathbf{E}_R = \mathbf{P} = 0$ [see (33)]. Therefore, like in the previous case, we have to find a timelike vector field and use the tensor \mathcal{F}_{ab} (6), which in this subcase has the following form

$$\mathcal{F}_{ab} = -\frac{\alpha^2 + \beta^2}{N} \left\{ \eta_{abcd} Q^c E_I^d + Q_a Q_b + E_{Ia} E_{Ib} - N g_{ab} \right\}.$$

Finally, the principal null directions are given by expressions (7). However, if we have a timelike vector orthogonal to \mathbf{E}_I and \mathbf{Q} , say \mathbf{U} ($U^a E_{Ia} = U^a Q_a = 0$), the principal directions can be found through the following expressions

$$K_a = \eta_{abcd} U^b Q^c E_I^d - N U_a, \quad L_a = \eta_{abcd} U^b Q^c E_I^d + N U_a.$$

3.2. Singular Papapetrou fields

As we have said before, this case is characterized by the vanishing of the complex function $\alpha + i\beta$, which is equivalent to the vanishing of the invariants of the Papapetrou field x and y (19):

$$\psi^a \psi_a = \omega^a \omega_a, \quad \psi^a \omega_a = 0. \quad (40)$$

Moreover, the exterior derivative of the Ernst potential ϵ (16) is a null 1-form, $\epsilon^a \epsilon_a = 0$, and the tensor \mathcal{F}_{ab} becomes $-T_{ab}$, that is, it is symmetric, in agreement with the fact that for singular 2-forms there is only one principal direction. It is obvious that in this case we cannot introduce the 1-form \mathbf{E} , but by virtue of (40), the exterior derivative of the Ernst potential ϵ is a complex 1-form satisfying similar properties that \mathbf{E} , in the sense that the real and imaginary parts of ϵ are also orthogonal.

In order to determine the principal direction of a singular Papapetrou field two subcases must be distinguished: (i) When ψ and ω are two spacelike 1-forms which are mutually orthogonal. (ii) The rest of possibilities, which are characterized by the fact that the 1-form ϵ , which is a null complex 1-form, is proportional to a real null 1-form \mathbf{k} .

3.2.1. Subcase (i): As we have said before, we consider ψ and ω to be spacelike and orthogonal. They cannot be timelike, since (40) would imply that they should be null 1-forms, which corresponds to the subcase (ii). On the other hand, either ξ or the 1-form ϕ must be timelike. Then, we can determine the principal null direction by contracting the energy-momentum tensor with one of them. We find that this principal direction is given by

$$\mathbf{k} = \psi^a \psi_a \xi - N \phi. \quad (41)$$

3.2.2. Subcase (ii): This case, characterized by

$$\epsilon = \lambda \mathbf{k}, \quad (42)$$

where λ is a complex function, includes some particular cases but in all of them ψ and ω are null or vanishing 1-forms. As we can see from expression (15), the null vector \mathbf{k} in (42) determines the principal null direction of the Papapetrou field. Finally, it is important to point out that in this case the principal null direction is orthogonal to the KVF ξ , $k^a \xi_a = 0$.

4. Some properties of the principal directions of the Papapetrou fields

We devote this section to study some properties and features of the principal null directions of the Papapetrou field in vacuum space-times. More specifically, we study under which conditions these null principal directions are geodesic, and in that case we compute the optical scalars associated with them. Here, we will also distinguish between the regular and singular cases.

4.1. Regular Papapetrou fields

First of all, we are going to study when a principal direction of the Papapetrou field is geodesic. We can do that by using the explicit expressions for the principal directions given in the previous section, or their properties in the case $\mathbf{E}_R = 0$, since in this case we do not know them explicitly. Then, the condition of geodesicity for a principal null direction is equivalent to say that the components of the vector $k^b k^a_{;b}$ on the orthogonal 2-planes must vanish. Then, contracting this vector with the explicit expressions for the generators of the orthogonal 2-planes (excepting in the case $\mathbf{E}_I = 0$, since in this case we do not have explicit expressions for them) and making some straightforward calculations, we can establish the conditions that a principal direction \mathbf{k} must satisfy in order to be geodesic. The main point in these calculations is the use of the following relation between \mathbf{P} , \mathbf{Q} and ξ

$$\mathbf{Q} - \mathbf{P} = \xi, \quad (43)$$

which follows from (30). This expression allows us to transform terms with \mathbf{Q} into terms with \mathbf{P} [and hence, into terms with the principal directions through (33)] and ξ . In this way, we can arrive at the following theorem:

Theorem 1: Let (V_4, g) be a vacuum space-time and let ξ be a KVF. Then, a principal direction \mathbf{k} of the Papapetrou field associated with ξ is geodesic if and only if the following condition holds

$$\begin{aligned} k^a k^b E_{a;b} &= 0 & \text{when } \mathbf{E}_I \neq 0, \\ m^a (\alpha + i\beta)_{,a} &= 0 & \text{when } \mathbf{E}_I = 0. \end{aligned}$$

Here, some remarks are in order. First, \mathbf{m} is an arbitrary complex 1-form lying on the orthogonal 2-planes and such that $\mathbf{m} \wedge \bar{\mathbf{m}} \neq 0$. We can find it by means of the tensor \mathcal{F}_{ab}^\perp (9,35). Second, when \mathbf{E}_R is a null 1-form we have seen that it is also one of the principal directions, and from expressions (37,38) we deduce that \mathbf{E}_R is in addition integrable, and therefore, it is geodesic. Finally, it is important to note that in the case $\mathbf{E}_I \neq 0$, the information on the geodesicity of a principal direction is encoded in a complex scalar depending only on the principal direction itself and the 1-form \mathbf{E} . In the case $\mathbf{E}_I = 0$, it depends on the gradient of $\alpha + i\beta$ on the orthogonal 2-planes.

When one of the principal directions is geodesic, we can study its optical scalars: expansion, shear and rotation (they are only well defined in the case of geodesic null vector fields). We begin with the shear of the geodesics tangent to a given principal direction. As is well known, if we choose the tangent vector field \mathbf{k} to be affinely parametrized, then the shear scalar coincides with the quantity $\sigma\bar{\sigma}$, where σ is one of the Newman-Penrose spin coefficients (see [7, 8]). But in general, we can take a Newman-Penrose basis associated with \mathbf{k} , namely $\{\mathbf{k}, \ell, \mathbf{m}, \bar{\mathbf{m}}\}$, and then to compute the shear tensor \mathcal{S}_{ab}

$$\mathcal{S}_{ab} \equiv H_a^c H_b^d k_{(c;d)} - \frac{1}{2} H_{ab} H^{cd} k_{c;d}, \quad H_{ab} = 2m_{(a}\bar{m}_{b)}, \quad H^a_a = 2, \quad (44)$$

where H_{ab} is the orthogonal projector to the 2-planes spanned by $\{\mathbf{k}, \ell\}$. Then, taking into account that the shear tensor \mathcal{S}_{ab} is spacelike, the shear scalar $\mathcal{S}^2 \equiv \frac{1}{2} \mathcal{S}^{ab} \mathcal{S}_{ab}$ vanishes if and only if \mathcal{S}_{ab} does, and it coincides with $\sigma\bar{\sigma}$ when \mathbf{k} is chosen to be affinely parametrized.

For our purposes it is better to compute the shear tensor (44). In the case $\mathbf{E}_I \neq 0$ we can take \mathbf{m} to be parallel to $\mathbf{E}_I + i\mathbf{Q}$ (with a proportional factor such that $m^a \bar{m}_a = 1$), whereas in the second case it can be found through the tensor \mathcal{F}_{ab}^\perp (9,35). Then, after some long but straightforward calculations and using here also the relation (43), we arrive to the following result

$$\mathcal{S}^2 = \mathcal{Z} \bar{\mathcal{Z}}, \quad \text{where} \quad \mathcal{Z} = \begin{cases} \frac{1}{E_I^c E_{Ic}} k^a E^b E_{[a;b]} & \text{if } \mathbf{E}_I \neq 0, \\ \frac{k^c E_{Rc}}{N} m^a m^b E_{Ra;b} = 0 & \text{if } \mathbf{E}_I = 0. \end{cases}$$

Then, using this result we can enounce the next theorem:

Theorem 2: Let (V_4, g) be a vacuum space-time and let ξ be a KVF. Then, a *geodesic* principal direction \mathbf{k} of the Papapetrou field associated with ξ is shear-free if and only if the following condition holds

$$\begin{aligned} k^a E^b E_{[a;b]} &= 0 & \text{when } \mathbf{E}_I \neq 0, \\ m^a m^b E_{Ra;b} &= 0 & \text{when } \mathbf{E}_I = 0. \end{aligned}$$

Here it is important to point out that the condition for the case $\mathbf{E}_I \neq 0$ involves only antisymmetric covariant derivatives and therefore, this calculation does not need the use of connection, we can make it by using only partial differentiation. In fact, it is possible to show that this condition can be written as follows

$$\left[(k^b E_b) E^a + N k^a \right] (\alpha + i\beta)_{,a} = 0$$

that is, in terms of derivatives of the eigenvalues of the Papapetrou field. Finally, taking into account the Goldberg-Sachs theorem [16], which states that a vacuum space-time is algebraically special if and only if it contains a shearfree geodesic congruence, the next corollary follows immediately

Corollary: Let (V_4, g) be a vacuum space-time and let ξ be a KVF. If one of the principal directions of the Papapetrou field associated with ξ , namely \mathbf{k} , satisfies the following conditions

$$\begin{aligned} k^a k^b E_{a;b} = 0 \quad \text{and} \quad k^a E^b E_{[a;b]} = 0 & \quad \text{when } \mathbf{E}_I \neq 0, \\ m^a (\alpha + i\beta)_{,a} = 0 \quad \text{and} \quad m^a m^b E_{Ra;b} = 0 & \quad \text{when } \mathbf{E}_I = 0, \end{aligned}$$

the space-time is algebraically special and \mathbf{k} is a principal direction of the Weyl tensor.

Now, let us study the expansion and rotation in the case of geodesic principal directions of the Papapetrou field. These quantities are given by

$$\vartheta \equiv \frac{1}{2} H^{ab} k_{a;b}, \quad H_a{}^c H_b{}^d k_{[c;d]} \equiv -2i\varpi m_{[a} \bar{m}_{b]},$$

where ϑ and ϖ are the expansion and rotation scalars respectively. In order to compute them it is more appropriate to consider the complex scalar

$$\rho \equiv -(\vartheta + i\varpi) = -m^a \bar{m}^b k_{a;b}, \quad (45)$$

usually called the *complex divergence* scalar, which is another spin coefficient in the Newman-Penrose formalism. The result of this calculation is

$$\rho = \begin{cases} \frac{1}{E_I^c E_{Ic}} (k^a E^b E_{[a;b]} + k^a E_R{}^b E_{a;b}) & \text{if } \mathbf{E}_I \neq 0, \\ -\frac{1}{N} [(k^c E_{Rc}) m^a \bar{m}^b E_{R(a;b)} + (k^c \xi_c) W_{ab} m^a \bar{m}^b] & \text{if } \mathbf{E}_I = 0. \end{cases}$$

In the case $\mathbf{E}_I = 0$ we can identify immediately the expansion and shear scalars, they are

$$\vartheta = \frac{k^c E_{Rc}}{N} m^a \bar{m}^b E_{R(a;b)}, \quad \varpi = -i \frac{k^c \xi_c}{N} W_{ab} m^a \bar{m}^b. \quad (46)$$

Here, it is interesting to note the direct relationship between the rotation W_{ab} of ξ and the rotation ϖ of the principal direction. With regard to the case $\mathbf{E}_I \neq 0$, it is remarkable that when the geodesic principal direction \mathbf{k} is also shear-free, the complex divergence is simply

$$\rho = \frac{1}{E_I^c E_{Ic}} k^a E_R{}^b E_{a;b},$$

and hence, the expansion and rotation scalars are given by

$$\vartheta = \frac{1}{E_I^c E_{Ic}} k^a E_R{}^b E_{Ra;b}, \quad \varpi = \frac{1}{E_I^c E_{Ic}} k^a E_R{}^b E_{Ia;b}.$$

Finally, it is important to remark the role that the 1-form \mathbf{E} plays also in the differential properties of the principal null directions of a regular Papapetrou field, since as we can see in all the expressions of this section, the optical scalars as well as the geodesicity condition are given only in terms of scalars formed from the principal direction \mathbf{k} and the covariant derivative of \mathbf{E} .

4.2. Singular Papapetrou fields

The general situation in the singular case can be summarized as follows: we have a vacuum space-time containing a singular 2-form (the Papapetrou field), which is a solution of the Maxwell equations (13). Therefore, the Mariot-Robinson theorem [21, 22] (see also [8]) tells us that this is equivalent to say that there is a geodesic and shear-free congruence of null curves. In addition, the principal null direction of the Papapetrou field will be tangent to these curves. Finally, the Mariot-Robinson theorem tell us that the space-time is algebraically special and then, the following theorem follows

Theorem 3: Let (V_4, g) be a vacuum space-time and let ξ be a KVF. If the gradient of the Ernst potential ϵ is a complex null 1-form, then the space-time will be algebraically special and the principal null direction of the

singular Papapetrou field associated with ξ will be geodesic and shear-free.

Taking into account that we are dealing with vacuum space-times, this theorem tells us that if the gradient of any KVF in a vacuum space-time is a null 1-form, then the space-time must be algebraically special and a multiple direction of the Weyl tensor coincides with the null principal direction of the corresponding Papapetrou field.

On the other hand, for the subcase (i), the complex divergence ρ (45) is given by the following expression

$$\rho = \frac{1}{2\psi^c\psi_c} k^a \bar{\epsilon}^b \epsilon_{b;a}.$$

In the subcase (ii), using the equation (17), we can see that $\rho = 0$.

5. Some examples

In this section we apply our previous study to some examples. In particular, we study the Papapetrou field associated with KVFs in the following vacuum space-times: (i) The Kerr metric. (ii) The Kasner metrics. (iii) A subclass of the plane-fronted gravitational waves. In these examples we compute the principal direction(s) of the Papapetrou field and also discuss some of their algebraic and differential properties.

In the first example we consider the well-known Kerr metrics [12]. They constitute a family of stationary axisymmetric vacuum solutions of Einstein's field equations. In Boyer-Lindquist (BL) coordinates $\{t, r, \theta, \varphi\}$ (see [23, 8]), the line-element takes the form

$$ds^2 = -\frac{\Delta}{\rho^2} (dt - a \sin^2 \theta d\varphi)^2 + \frac{\sin^2 \theta}{\rho^2} [(r^2 + a^2) d\varphi - a dt]^2 + \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2, \quad (47)$$

where Δ and ρ are given by $\Delta = r^2 - 2Mr + a^2$ and $\rho^2 = r^2 + a^2 \cos^2 \theta$ respectively. Here, we must say that in the case $a^2 < M^2$, the BL coordinates are valid in the asymptotically flat regions ($r_+ < r < \infty$) or type I regions, in the type II regions ($r_- < r < r_+$), which contain closed trapped surfaces, and in the asymptotically flat regions containing the ring singularity $\rho^2 = 0$ ($-\infty < r < r_-$) or type III regions ($r_{\pm} = M \pm (M^2 - a^2)^{1/2}$; see [18] for more details). The only singularity of this metric is the ring singularity $\rho^2 = 0$. The event horizons ($\Delta = 0 \Leftrightarrow r = r_{\pm}$) and the axis of symmetry ($\theta = 0$) are just coordinate singularities, as is well-known, we can avoid them by choosing other coordinate systems.

This metric has an Abelian two-parameter group of isometries. We can choose two independent KVF's in the following way

$$\vec{\xi}_t = \frac{\partial}{\partial t}, \quad \xi_t = - \left(1 - \frac{2Mr}{\rho^2} \right) dt - \frac{2Mar \sin^2 \theta}{\rho^2} d\varphi, \quad (48)$$

$$\vec{\xi}_{\varphi} = \frac{\partial}{\partial \varphi}, \quad \xi_{\varphi} = - \frac{2Mar \sin^2 \theta}{\rho^2} dt + \frac{\Sigma^2 \sin^2 \theta}{\rho^2} d\varphi, \quad (49)$$

where $\Sigma^2 \equiv (r^2 + a^2)^2 - a^2 \Delta \sin^2 \theta$. Now, we will study the Papapetrou field associated with the timelike KVF, ξ_t ,

$$\begin{aligned} F &= \frac{2M}{\rho^4} (r^2 - a^2 \cos^2 \theta) (dt - a \sin^2 \theta d\varphi) \wedge dr \\ &+ \frac{2Mar \sin(2\theta)}{\rho^4} d\theta \wedge (adt - (r^2 + a^2) d\varphi). \end{aligned} \quad (50)$$

It is important to point out (see [4]) that this Papapetrou field is proportional to the electromagnetic field of the Kerr-Newman metric (see also [5] for a detailed study of this electromagnetic field). As we can see from (48), the norm of ξ_t is given by

$$N = - \left(1 - \frac{2Mr}{\rho^2} \right) = - \frac{\Delta - a^2 \sin^2 \theta}{\rho^2}. \quad (51)$$

As is clear, it vanishes on the surfaces of infinite redshift ($\Delta = a^2 \sin^2 \theta$), and in principle we cannot apply our procedure there. However, this situation is different from that of Appendix A, and later we will see that it is possible to use the general procedure of Sec. 2.

In the regions where the BL coordinates are valid, we find the following expressions for ψ and ω

$$\begin{aligned} \psi &= \frac{2M}{\rho^4} \{ (r^2 - a^2 \cos^2 \theta) dr - a^2 \sin(2\theta) r d\theta \}, \\ \omega &= \frac{2Ma \sin \theta}{\rho^4} \{ \cot \theta dr + (r^2 - a^2 \cos^2 \theta) d\theta \}, \end{aligned}$$

and then, we can write the Ernst potential as follows

$$\mathcal{E} = - \frac{2M}{\rho^2} (r + ia \cos \theta) = - \frac{2M}{r - ia \cos \theta}.$$

The eigenvalues of the Papapetrou field (50) are given by

$$\alpha + i\beta = s \frac{2M}{(r - ia \cos \theta)^2} = s \frac{\mathcal{E}^2}{2M}, \quad (52)$$

and therefore, it is regular everywhere excepting in the ring singularity $\rho^2 = 0$, where it diverges. On the other hand, it is remarkable that $\alpha + i\beta$ is an analytic function of the Ernst potential. As we have pointed out before, this can only happens when $d\mathbf{E} = 0$ [see equation (23)], like in this example, where the 1-form \mathbf{E} is given by

$$\mathbf{E} = s(d\mathbf{r} + ia \sin \theta d\theta) \implies d\mathbf{E} = 0.$$

Moreover,

$$\mathbf{E}_R = s d\mathbf{r}, \quad E_R^a E_{Ra} = \frac{\Delta}{\rho^2}, \quad \mathbf{E}_I = sa \sin \theta d\theta, \quad E_I^a E_{Ia} = \frac{a^2 \sin^2 \theta}{\rho^2},$$

and hence, excepting in $r = r_{\pm}$ ($\Delta = 0$) and in the axis of symmetry ($\theta = 0$), the Papapetrou field corresponds to the generic case (Subsection 3.1.1). In order to compute the principal directions of the Papapetrou field (50) in the generic case, we need to compute the 1-form \mathbf{D}

$$\mathbf{D} = -\frac{a\Delta \sin^2 \theta}{\rho^2} d\varphi, \quad D^a D_a = \frac{a^2 \Delta \sin^2 \theta (\Delta - a^2 \sin^2 \theta)}{\rho^6}.$$

Then, \mathbf{P} and \mathbf{Q} are

$$\mathbf{P} = \frac{\Delta}{\rho^2} (d\mathbf{t} - a \sin^2 \theta d\varphi) \quad \mathbf{Q} = \frac{a \sin^2 \theta}{\rho^2} (a d\mathbf{t} - (r^2 + a^2) d\varphi),$$

and finally, the principal null directions of the Papapetrou field are [see (33)]

$$\mathbf{K} = -\frac{\Delta}{\rho^2} d\mathbf{t} + s d\mathbf{r} + \frac{a \sin^2 \theta \Delta}{\rho^2} d\varphi, \quad \mathbf{L} = -\frac{\Delta}{\rho^2} d\mathbf{t} - s d\mathbf{r} + \frac{a \sin^2 \theta \Delta}{\rho^2} d\varphi, \quad (53)$$

which are just the multiple principal directions[¶] of the Kerr space-time (see for instance [6]), whose Petrov type is D. We can check, using Theorems 1 and 2, that \mathbf{K} and \mathbf{L} are indeed geodesic and shearfree.

On the other hand, in Appendix B we have given the connection with the *eigenray* formalism of Perjés, showing that the existence of a geodesic (and shear-free) principal direction of the Papapetrou field is equivalent to the existence of a geodesic (and shear-free) eigenray. This result, together with the fact that the Kerr space-time is the only asymptotically-flat stationary vacuum solution with geodesic and shear-free eigenrays [15], shows that the Kerr metric is the only one with such characteristics in which the alignment of the principal directions of the Papapetrou and gravitational fields takes place, which has been shown recently by Mars [14]. In this paper, the characterization is given in the whole Kerr manifold, whereas the result of Perjés [15] was found only in the region where the KVF is timelike. However, taking into account that the eigenray formalism can be used also in the case of spacelike KVFs, Perjés' result could be extended to the whole manifold.

[¶] Note that the arbitrary sign s only serves to change from \mathbf{K} to \mathbf{L} and the converse.

Now, in order to show the capabilities of our formalism, we will extend the analysis to the symmetry axis, the even horizons, and the surfaces of infinite redshift. On the axis of symmetry (where $\mathbf{E}_I = 0$ and $E_R^a E_{Ra} \neq 0$) we can find the principal directions by using expressions (34). However, the BL coordinates are not good on the axis. In order to solve this problem we can make the calculation in Kerr-Schild coordinates (see [8, 18]). It is easy to see that the principal directions obtained can be expressed in terms of the coordinates $\{t, r\}$, they are given by

$$\mathbf{K} = - \left(1 - \frac{2Mr}{r^2 + a^2} \right) dt + s dr, \quad \mathbf{L} = - \left(1 - \frac{2Mr}{r^2 + a^2} \right) dt - s dr.$$

The surfaces of infinite redshift ($\Delta = a^2 \sin^2 \theta$) constitute a controversial point in this example, because there the norm of ξ_t vanishes [see (51)]. However, $\alpha + i\beta \neq 0$ (52) is different from zero on these surfaces, which tells us that we are not in the case of Appendix A. Then, we can follow the general procedure explained in Sec. 2. In this case, the Papapetrou field belongs to the generic case, and we can see that the principal directions are given by (53).

In the case of the event horizons, we are going to consider the maximal extension of the Kerr metric. As is well-known, it can be constructed by using advanced and retarded Kerr coordinates $\{V_\pm, r, \theta, \varphi_\pm\}$ ($-\infty < V_\pm < \infty$, $-\infty < r < \infty$, $0 \leq \theta < \pi$, $0 \leq \varphi_\pm < 2\pi$)

$$dV_\pm = dt \pm (r^2 + a^2)\Delta^{-1}dr, \quad d\varphi_\pm = d\varphi \pm a\Delta^{-1}dr.$$

Now, the line element is analytic (apart from regions I, II, and III) also in $r = r_\pm$. Therefore, we can already compute the principal directions on r_\pm , where we have $E_R^a E_{Ra} = 0$ and $\mathbf{E}_R \neq 0$. Taking into account the procedure explained in Sec. 3, one of the principal directions is given by $\mathbf{E}_R = d\mathbf{r}$, and the other one is obtained by contracting (6) with any timelike vector field. In our case, we find that on the event horizons \mathcal{F} is given by

$$\mathcal{F} = 2(dV_\pm - a \sin^2 \theta d\varphi_\pm) \otimes d\mathbf{r},$$

and therefore, the other principal direction is given by $dV_\pm - a \sin^2 \theta d\varphi_\pm$ (the vector field associated with this 1-form is $\partial/\partial r$, which is tangent to the curves $V_\pm = \text{constant}$).

To sum up, the maximal extension of the Kerr metric (see [18] for details), in the case $a^2 < M^2$, is made up of an infinite chain of asymptotically-flat type I regions, connected to type II regions (which contain trapped surfaces) and asymptotically-flat type III regions (which contain the ring singularity) by means of the null hypersurfaces $r = r_\pm$. We have just seen that we can compute the principal directions of the Papapetrou field (50) for the whole Kerr manifold, and in addition, that they coincide with the repeated principal directions of the Weyl tensor. For the sake of completeness, we want to remark that these results can be trivially extended to the cases $a^2 = M^2$ (where $r_+ = r_-$ and there is no region II) and $a^2 > M^2$ (where $\Delta > 0$ everywhere).

We finish this example with a comment on the other KVF, ξ_φ (49). Both KVFs are privileged since ξ_t is the only one (up to a multiplicative constant) which is timelike at

arbitrary large positive and negative values of r , and ξ_φ is the only one which vanishes on the axis of symmetry ($\theta = 0$) and satisfies the regularity condition there (see [8]). However, we can check that the principal directions of the Papapetrou field associated with ξ_φ do not coincide with the principal directions of the Weyl tensor. In this sense, our study privileges the KVF ξ_t over the KVF ξ_φ .

The next example will be an algebraically general vacuum space-time. One of the most simple families of this type is the following class of Kasner metrics [24] (see also [8])

$$ds^2 = -dt^2 + t^{2p_1}dx^2 + t^{2p_2}dy^2 + t^{2p_3}dz^2,$$

where p_1 , p_2 and p_3 are constants satisfying the following relationships

$$p_1 + p_2 + p_3 = 1, \quad p_1^2 + p_2^2 + p_3^2 = 1.$$

This metric has an Abelian G_3 group of isometries acting on the spacelike hypersurfaces $\{t = \text{constant}\}$. We can choose one of these KVFs, for instance

$$\vec{\xi}_x \equiv \frac{\partial}{\partial x}, \quad \xi_x = t^{2p_1} dx \quad \implies \quad F = 2p_1 t^{2p_1} dt \wedge dx.$$

Then,

$$N = t^{2p_1}, \quad \psi = -2p_1 t^{2p_1-1} dt, \quad \omega = 0,$$

which means that the Ernst potential associated with ξ_x is simply $\mathcal{E} = -t^{2p_1}$. Now, if we compute the eigenvalues of the Papapetrou field

$$\alpha = 2sp_1 t^{p_1-1}, \quad \beta = 0,$$

we realize that it is a regular electromagnetic field, but it does not belong to the generic class, since

$$E = -st^{p_1} dt \quad \iff \quad E_R = -st^{p_1} dt, \quad E_I = 0.$$

That is to say, we are in the particular case $E_I = 0$, and therefore the principal directions of the Papapetrou field are given by (34)

$$\vec{K} = st^{p_1} \frac{\partial}{\partial t} + \frac{\partial}{\partial x}, \quad \vec{L} = -st^{p_1} \frac{\partial}{\partial t} + \frac{\partial}{\partial x}. \quad (54)$$

The complex vector \mathbf{m} which determines the orthogonal 2-planes to the principal directions, which is obtained from (9) or (35), can be taken as follows

$$\vec{m} = \frac{1}{\sqrt{2}} \left(t^{-p_2} \frac{\partial}{\partial y} + it^{-p_3} \frac{\partial}{\partial z} \right).$$

Now, we can check that the null principal directions (54) of the Papapetrou field satisfy the condition of the Theorem 1, and therefore they are geodesic. In the general case, they do not coincide with the Weyl principal directions. Moreover, they satisfy the condition of the Theorem 2 only when $p_2 = p_3$, in which case they are also shearfree and hence, via the Goldberg-Sachs theorem [16], the space-time is algebraically special

(specifically, Petrov type D), being (54) the repeated null principal directions. Finally, using expressions (46), we find that the expansion of \mathbf{k} and \mathbf{l} is given by

$$\vartheta_{\pm} = \pm \frac{1}{2} s(p_1 - 1)t^{p_1-3},$$

(the signs $+$ and $-$ are for \mathbf{k} and \mathbf{l} respectively), and the rotation is zero since the rotation of ξ_x also vanishes [see (46)].

Another interesting example is a subfamily of the plane-fronted gravitational waves, *pp waves* (see for instance [8]). The line-element of these space-times is

$$ds^2 = -2dudv + 2d\zeta d\bar{\zeta} - 2Hdu^2,$$

where

$$H = f(\zeta) + \bar{f}(\bar{\zeta}),$$

that is, this is the subclass of the vacuum plane waves with the additional KVF

$$\vec{\xi}_u = \frac{\partial}{\partial u}, \quad \xi_u = -dv - 2Hdu, \quad N = -2H = -2(f(\zeta) + \bar{f}(\bar{\zeta})).$$

(These metrics have always $\partial/\partial v$ as a KVF but it is a null KVF; see Appendix A). The Papapetrou field associated with ξ_u is given by

$$\mathbf{F} = 2d\mathbf{u} \wedge (H_{,\zeta}d\zeta + H_{,\bar{\zeta}}d\bar{\zeta}), \quad (55)$$

Then, the 1-forms ψ and ω are given by

$$\psi = 2(H_{,\zeta}d\zeta + H_{,\bar{\zeta}}d\bar{\zeta}), \quad \omega = -2i(H_{,\zeta}d\zeta - H_{,\bar{\zeta}}d\bar{\zeta}). \quad (56)$$

From these expressions we can write the Ernst potential associated with ξ_u as follows

$$\mathcal{E} = 4f(\zeta).$$

Furthermore, from (56) we deduce that

$$\psi^a \psi_a = \omega^a \omega_a = 8H_{,\zeta}H_{,\bar{\zeta}} \geq 0, \quad \psi^a \omega_a = 0.$$

Then, the Papapetrou field (55) is singular ($\alpha + i\beta = 0$), and it corresponds to the subcase (i) of the subsection 3.2. Using (41), the principal null direction is given by

$$\vec{\mathbf{k}} = 16HH_{,\zeta}H_{,\bar{\zeta}}\frac{\partial}{\partial v}, \quad (57)$$

which is the propagating direction of the plane wave, orthogonal to the wave fronts. Moreover, it is proportional to the null KVF $\partial/\partial v$ and hence, taking into account the Theorem 3, \mathbf{k} is geodesic and shear-free. From the Goldberg-Sachs theorem [16], we deduce that it is also a principal null direction of the space-time. In this case the metric is of the Petrov type N, with only one multiple principal direction, which is given by (57). Finally, the complex divergence ρ vanishes since $\partial/\partial v$ is a constant vector field.

6. Remarks and conclusions

In this paper we have developed a formalism for the study of the Papapetrou fields in vacuum space-times. In the first part of this paper, we have determined the null principal direction(s). In the case of regular Papapetrou fields, this is very useful in order to solve Maxwell's equations in curved space-times, since when we write them in a Newman-Penrose basis adapted to these principal directions the conditional differential system for them is second-order, contrary to what happens when we write them in an arbitrary Newman-Penrose basis, in which case it is a third-order differential system. In the second part of this work, we have studied the main differential properties of the null principal direction(s) of the Papapetrou field. Specifically, we have found the condition that a principal direction must satisfy in order to be geodesic. Moreover, for geodesic principal directions we have found explicit expressions for the optical scalars. This study allows us to study when a principal direction of the Papapetrou field is also a principal direction of an algebraically special vacuum space-time. Furthermore, taking into account the simplicity of the expressions for the differential properties, and the fact that they only depend on the derivatives of the Ernst potential and the principal direction, they provide an interesting way of introducing new *Ansätze* for the search of exact solutions. The examples studied here can serve as a guide. As we have seen, the principal directions of the Kerr space-time are aligned with those of Papapetrou field associated with the timelike KVF (see also [13, 14]), which gives another characterization of this metric [14], and the other two cases (Kasner and *pp waves*) are also examples of how the algebraic structure of the Papapetrou field can be adapted (for some KVFs) to the algebraic structure of the gravitational field. In this sense, the expressions for the expansion and rotation scalars of the principal directions are also useful.

Finally, this study can be extended in several ways. One of them is to study the relationship between the principal direction(s) of the Papapetrou fields and the possible algebraic types of the space-time. Here, there are several cases depending on the multiplicity and degree of alignment of the principal direction(s) of the Papapetrou field [25]. On the other hand, taking into account that most of the properties studied can be expressed in terms of objects related directly to the Ernst potential, it seems reasonable to extend this study to other space-times in which the Ernst potential can be defined, for instance to Einstein-Maxwell space-times possessing a Killing vector field.

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Appendix A. The case of a null Killing vector field

In this Appendix we sum up briefly what happens when we consider a null KVF. In order to study this case, it is enough to remember that all the vacuum space-times admitting a null Killing vector field are known, they can be found in [8] (section 21.4; see also Ref. [19]). The Killing equations (10) tell us that the KVF is a geodesic, shear-free, and expansion-free null vector field, and the Einstein field equations imply the vanishing of the twist $\xi \wedge d\xi = 0$. As it was shown by Dautcourt [26], these space-times can be divided into two different classes:

-Class I is the family of the *pp* waves (see [8]), in which the KVF is a constant null vector field ($\xi_{a,b} = 0$), and therefore the associated Papapetrou field is identically zero.

-Class II is determined by the following line element (see [8])

$$ds^2 = -2xdu(dv + Mdu) + x^{-1/2}(dx^2 + dy^2),$$

where the function $M = M(u, x, y)$ satisfies the following partial differential equation:

$$(xM_{,x})_{,x} + xM_{,yy} = 0.$$

In this case, the null KVF is

$$\vec{\xi} = \frac{\partial}{\partial v}, \quad \xi = -xdu. \quad (\text{A.1})$$

Then, the associated Papapetrou field is simply $F = du \wedge dx$. As is clear, it is singular and the corresponding principal direction is given by the null KVF (A.1). Therefore, the principal null direction is geodesic and shear-free, and if we choose the KVF (A.1) to be the tangent vector field, the integral curves are affinely parametrized, being v an affine parameter.

Appendix B. Connection with the *eigenray* formalism

In reference [15] Perjés developed a spinor calculus for stationary space-times from which, a triad formalism similar to the Newman-Penrose formalism [7] was put forward. An applications of this technique is the search of exact solutions of Einstein's equations. To that end, Perjés introduced the concept of *eigenray*. Here, we show the connection between this concept and our development.

Let us consider a non-null KVF ξ and a null vector field k normalized by $k^a \xi_a = 1$. These two objects determine a unit vector field n by

$$n = \sqrt{|N|} (k - N^{-1} \xi), \quad n^a n_a = -\text{sgn}(N), \quad (\text{B.1})$$

where $|x|$ and $\text{sgn}(x)$ denote the absolute value and the sign of x respectively. Now, we can show [15] that n is tangent to geodesics in the 3-space of the Killing orbits, which are called *eigenrays*, when it satisfies the following algebraic condition

$$p^{ab} \psi_b + \varepsilon^a_{bc} n^b \omega^c = 0, \quad \varepsilon_{abc} \equiv (|N|)^{-1/2} \eta_{dabc} \xi^d, \quad (\text{B.2})$$

where $p_{ab} \equiv g_{ab} + \text{sgn}(N) n_a n_b$ is the orthogonal projector to n . It can be also proven that condition (B.2) is equivalent to say that k is geodesic (see [15]), therefore n is

geodesic if and only if \mathbf{k} is geodesic. Moreover, Perjés showed that a geodesic eigenray is also shear-free if and only if it is the null vector field associated through (B.1).

Now, let us consider the Papapetrou field associated with ξ . It is possible to show that the null vector field \mathbf{k} in (B.1) is a principal direction of the Papapetrou field if and only if condition (B.2) holds. Therefore, this result shows that the existence of a geodesic eigenray is equivalent to the existence of a geodesic principal direction of the Papapetrou field (and the same for the shear-free case).

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